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Dynamics of Solitons in Inhomogeneous Josephson Junctions

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We examine the dynamics of sine-Gordon solitons in an inhomogeneous Josephson junction. Two types of inhomogeneity are worked out: (1) varying "refractive index" or dielectric constant, and (2) varying parameters in the nonlinear term. Simplified analytical explanations are also presented for the numerical results.

1. INTRODUCTION

Superconducting Josephson junctions (Josephson, 1962) exhibit soliton behavior which can be described by sine-Gordon equation [see, e.g., Pederson (1986), which we mainly follow in this introduction]. In this case solitons are quantums of magnetic flux. Their production, transmission, and storage as stable objects is quite feasible, and therefore very important in information processing systems.

The tunneling effect of Cooper pairs across a thin insulator between two superconductors was predicted by Josephson (1962). If the common macroscopic wave function of all the electron pairs is written as

$$\Psi = \sqrt{R}e^{i\phi} \tag{1}$$

the two superconductors will naturally have independent wave functions Ψ_1 and Ψ_2 with uncorrelated phases ϕ_1 and ϕ_2, unless the two superconductors are set near enough to each other (say less than about 30 Å). The phases then become correlated because of Cooper pair penetration across the insulator barrier.

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The wave functions Ψ_1 and Ψ_2 satisfy two coupled linear Schrödinger equations (Feynman *et al.*, 1965)

$$i\hbar \frac{\partial \Psi_1}{\partial t} = E_1\Psi_1 + k\Psi_2 \tag{2}$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = E_2\Psi_2 + k\Psi_1 \tag{3}$$

where E_1 and E_2 are the ground-state energies of electrons in the two superconductors. Here, we have assumed that the two superconductors are similar. k is a real coupling constant which depends on the characteristics of the junction. Obviously, $k \to 0$ as $d \to \infty$, where d is the barrier thickness. When a static potential difference V is maintained between the two superconductors, an energy shift $E_1 - E_2 = 2eV$ is developed. We can arbitrarily choose the reference energy at $E = (E_1 + E_2)/2 = 0$, and therefore $E_1 = eV$ and $E_2 = -eV$. Equations (2) and (3) then become

$$i\hbar \frac{\partial \Psi_1}{\partial t} = eV\Psi_1 + k\Psi_2 \tag{4}$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = -eV\Psi_2 + k\Psi_1 \tag{5}$$

Using the expressions $\Psi_1 = \sqrt{R_1}e^{i\phi_1}$ and $\Psi_2 = \sqrt{R_2}e^{i\phi_2}$ in these equations and separating the real and imaginary parts, we obtain

$$\hbar\, \partial R_1/\partial t = -2k(R_1 R_2)^{1/2} \sin \phi \tag{6}$$

$$\hbar\, \partial R_2/\partial t = +2k(R_1 R_2)^{1/2} \sin \phi \tag{7}$$

$$\hbar\, \partial \phi_1/\partial t = k(R_2/R_1)^{1/2} \cos \phi - eV \tag{8}$$

$$\hbar\, \partial \phi_2/\partial t = k(R_1/R_2)^{1/2} \cos \phi + eV \tag{9}$$

in which $\phi = \phi_2 - \phi_1$ is the phase difference between the two wave functions. Let us define the quantities $J_1 \equiv \partial R_1/\partial t$ and $J_2 \equiv \partial R_2/\partial t$. Here R_1 and R_2 represent electron pair densities which deviate only slightly from their equilibrium values R_0. We therefore have $R_1 \simeq R_2 \simeq R_0$, and $(2k/\hbar)(R_1 R_2)^{1/2} \simeq 2kR_0/\hbar \equiv J_0$, and therefore

$$J \simeq J_0 \sin \phi \tag{10}$$

according to (6) or (7).

Equations (8) and (9) therefore yield

$$\hbar \frac{\partial \phi}{\partial t} = 2eV \tag{11}$$

We can write equation (11) in the form

$$\frac{d\Phi}{dt} = V \tag{12}$$

where Φ has the dimensions of magnetic flux, and is defined according to

$$\phi = 2\pi \frac{\Phi}{\Phi_0} \tag{13}$$

in which $\Phi_0 \equiv h/2e$ is the quantum of magnetic flux. From (10) and (13) we have

$$\Phi = \frac{\Phi_0}{2\pi} \sin^{-1} \frac{J}{J_0} \tag{14}$$

If $V = 0$, then (13) implies $\Phi = \text{const}$, which is in general nonvanishing. This leads to a finite current density J even in the absence of an applied voltage. The effect is known as the dc Josephson effect. If $V = V_0 = \text{const}$, $\Phi = V_0 t + \Phi_1$ and (15) yields an alternating current density (ac Josephson effect)

$$J = J_0 \sin \frac{2\pi}{\Phi_0} (V_0 t + \Phi_1) \tag{15}$$

Therefore, an alternating current density develops with an angular frequency

$$\omega_J = \frac{2\pi V_0}{\Phi_0} = \frac{2eV_0}{\hbar} \tag{16}$$

We now turn to a long Josephson junction, which consists of two relatively long strips of superconducting materials separated by a very thin dielectric of thickness d. It can be shown that a length element dx of this device is electrically equivalent to the circuit shown in Fig. 1. Capacitance per unit length is

$$C = \frac{K\epsilon_0 a}{d} \tag{17}$$

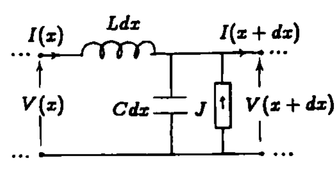

Fig. 1. Electrical equivalent of long Josephson junction.

in which K is the dielectric constant of the dielectric, $\epsilon_0 \approx 8.854 \times 10^{-12}$ F/m, and a is the width of the superconducting strip. Inductance per unit length is

$$L = \mu_0 \frac{2\lambda_L + d}{a} \tag{18}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}, and λ_L is the penetration depth of the superconductors.

The following equations result from basic circuit theory:

$$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} \tag{19}$$

$$\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t} - J_0 \sin 2\pi \frac{\Phi}{\Phi_0} \tag{20}$$

$$\frac{\partial \Phi}{\partial t} = V \tag{21}$$

These equations can be easily combined to yield the following sine-Gordon equation for the phase difference:

$$\frac{\partial^2 \phi}{\partial t^2} - c_J^2 \frac{\partial^2 \phi}{\partial x^2} + \omega_p^2 \sin \phi = 0 \tag{22}$$

in which

$$c_J = \frac{1}{(LC)^{1/2}} \qquad \text{and} \qquad \omega_p = \left(\frac{2\pi J_0}{\Phi_0 c}\right)^{1/2} \tag{23}$$

and (14) has been used. Note that c_J/ω_p has dimensions of length. It describes a length scale (called the Josephson penetration length), which when compared with the length of the junction determines whether a Josephson junction is 'long' or not. Equation (22) can obviously have the kink solution [see equation (24) in Section 2], in which $b = 1$, $a = \omega_p^2/c_J^2$, and $c \to c_J$. The corresponding voltage V and current I can then be easily calculated using equations (20) and (21). The kink (antikink) describes a pulse of 2π (-2π) phase difference, corresponding to a quantum of magnetic flux accompanied by a voltage and current pulse. The kink (antikink) is thus called a fluxon (antifluxon) in this case. Any spatial variation in the dielectric constant K results in a position-dependent c_J. This in turn affects the propagation of kinks in the junction. The situation can be approximated by the classical analog of a point particle moving in a (velocity-dependent) external potential. We will discuss this further in Section 3.

2. SINE-GORDON EQUATION

The standard form of the sine-Gordon equation is

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = a \sin b\phi \tag{24}$$

in which a and b are constants assumed to have the same sign. By using the change of variables

$$u = (ab)^{1/2}x; \qquad v = (ab)^{1/2}ct; \qquad \sigma = b\phi \tag{25}$$

equation (24) becomes

$$\sigma_{uu} - \sigma_{vv} = \sin \sigma \tag{26}$$

Multisoliton solutions of this equation can be obtained systematically by applying the Bäcklund transformation (Bäcklund, 1876). In this section, we will obtain kink solutions via an elementary separation-of-variables method (Lamb, 1980). Using the ansatz

$$\sigma(u, v) = 4 \tan^{-1} \frac{U(u)}{V(v)} \tag{27}$$

and the trigonometric identity

$$\sin \sigma = \frac{4 \tan(\sigma/4)[1 - \tan^2(\sigma/4)]}{1 + \tan^2(\sigma/4)} \tag{28}$$

we obtain

$$(U^2 + V^2)\left(\frac{U''}{U} + \frac{V''}{V}\right) - 2(U')^2 - 2(V')^2 = V^2 - U^2 \tag{29}$$

in which the primes indicate differentiation of the functions U and V with respect to their arguments. By differentiating (29) once with respect to u and once with respect to v, we can separate this equation into

$$\frac{1}{UU'}\left(\frac{U''}{U}\right)' = -\frac{1}{VV'}\left(\frac{V''}{V}\right)' = -4k^2 \tag{30}$$

These equations can now be easily twice integrated to yield

$$(U')^2 = -k^2U^4 + m^2U^2 + n^2 \tag{31}$$

$$(V')^2 = K^2V^4 + (m^2 - 1)V^2 - n^2 \tag{32}$$

in which m and n are integration constants. Solutions of these equations involve—in general—elliptic functions (Steuerwald, 1936). There are a few special cases which can yield simple yet important soliton solutions. Let $k = 0$, $m > 1$, and $n = 0$ in (31) and (32). These equations can immediately be integrated to yield $U(u) = \gamma_1 \exp(\pm mu)$ and $V(v) = \gamma_2 \exp[\pm(m^2 - 1)^{1/2}v]$, in which γ_1 and γ_2 are constants of integration. Therefore

$$\sigma(u, v) = 4 \tan^{-1}\Gamma \exp[\pm mu \pm (m^2 - 1)^{1/2}v] \tag{33}$$

Note that all sign combinations are possible. Using (25) and the $(+, -)$ choice of signs, we obtain

$$\phi(x, t) = \frac{4}{b} \tan^{-1}\{\Gamma \exp[(ab)^{1/2}\gamma(v_0)(x - v_0t)]\} \tag{34}$$

which is known as the kink solution. In this equation, $\gamma(v_0) = (1 - v_0^2/c^2)^{-1/2}$ and $v_0 = (1 - 1/m^2)^{1/2}$.

3. KINK DYNAMICS IN AN INHOMOGENEOUS MEDIUM

In this section we examine the dynamics of sine-Gordon kinks in an inhomogeneous medium. The time-independent inhomogeneity can be introduced into the sine-Gordon equation in different ways. We will mention three ways, although only two cases will be worked out in detail.

The first kind of inhomogeneity is introduced via a varying 'refractive index':

$$\frac{\partial^2\phi}{\partial x^2} - \frac{n^2(x)}{c^2}\frac{\partial^2\phi}{\partial t^2} = a \sin b\phi \tag{35}$$

This looks like a variation in the optical refractive index of a transparent medium in the context of electromagnetic wave propagation. A physical fulfillment of equation (35) can be accomplished by an inhomogeneous Josephson transmission line.

Consider equation (22). If we let the dielectric constant K vary with x, we can write

$$c_J = \frac{\bar{c}}{n(x)} \tag{36}$$

where

$$\bar{c} = \frac{1}{(LC_0)^{1/2}} \tag{37}$$

and

$$n(x) = \left(\frac{K(x)}{K_0}\right)^{1/2} \tag{38}$$

In equation (37), C_0 is a constant reference capacitance corresponding to the dielectric constant K_0.

The inhomogeneity of the second kind can be introduced via spatially varying a and b:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = a(x)\sin b(x)\phi \tag{39}$$

This kind of inhomogeneity will be discussed in detail in Section 3.2. The third kind of inhomogeneity is introduced via an external field $\chi(x)$:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = a\sin b\phi + \chi(x) \tag{40}$$

This case is discussed in Reinisch and Fernandez (1981) and Kaup (1984). It seems less interesting to us, and therefore we shall not consider it any further.

Several interesting questions arise when the kink dynamics in an inhomogeneous medium is concerned. If the kink is considered as a classical particle interacting with a background potential (Riazi, 1993), what would be the characteristics of such a potential in relation to $n(x)$, $a(x)$, $b(x)$, and the kink velocity v_k? In what circumstances does the particle aspect fail to be adequate? What are the interesting features of the wave aspect, etc.?

Although interesting analytical approximations can be worked out which explain some of the basic properties of the kink dynamics, a more elaborate picture of what is going on can only be achieved via numerical integration. The numerical procedure we have followed in order to carry out the integration is as follows. The x axis in the relevant range is divided into N divisions of length ϵ. The time axis is also divided into intervals of duration δ. Using the standard finite-difference expressions for spatial and temporal derivatives, we can show that

$$\sigma_{i,j+1} = \frac{\delta^2(\sigma_{i+1,j} - 2\sigma_{i,j} + \sigma_{i-1,j} - \epsilon^2 a_i \sin(b_i\sigma_{i,j}))}{\epsilon^2 n_i^2} + 2\sigma_{i,j} - \sigma_{i,j-1} \tag{41}$$

where $\sigma_{i,j} \equiv \sigma(u = i\epsilon, v = j\delta)$, $a_i = a(u = i\epsilon)/a_0$, $b_i = b(u = i\epsilon)/b_0$, and $n_i = n(u = i\epsilon)$. As before, $\sigma = b_0\phi$, $u = (a_0b_0)^{1/2}x$, and $v = (a_0b_0)^{1/2}ct$, with a_0 and b_0 being some reference values for a and b, respectively. Equation (41) enables us to calculate the field configuration in a next time step, using its configuration at two preceding time steps. Starting from an initial configuration at time steps $j = 1$ and $j = 2$, equation (41) can be successively applied to calculate the kink dynamics up to any arbitrary later time.

3.1. Inhomogeneity of the First Kind

Kink dynamics in a medium with $n(x)$ can be approximated in terms of a classical particle moving against a velocity-dependent potential. Consider a kink with initial velocity v_0 incident on a potential barrier

$$n(x) = \begin{cases} 1.0 & \text{if} \quad x < x_1 \\ n_1 > 1 & \text{if} \quad x \geq x_1 \end{cases}$$

Numerical results indicate that low-velocity kinks do penetrate the barrier. There is a threshold velocity above which kinds cannot penetrate the barrier. This threshold velocity depends of course on the potential height n_1. Ultrafast kinks cause kink–antikink pair production together with low-amplitude excitations. Examples are shown in Fig. 2 for $n_1 = 1.02$ and $v_k = 0.01$, 0.1, and 0.97. The velocity dependence of the force acting on the kink is evident also in Fig. 3, where a kink moves across a linearly increasing refractive index

$$n(x) = \begin{cases} 1.0 & \text{if} \quad x < x_1 \\ 1 + \alpha(x - x_1) & \text{if} \quad x \geq x_1 \end{cases}$$

The kink is observed to penetrate to a certain depth, where it is almost frozen. It will never come back to recover its 'potential energy.' A simple analytical description can be presented in the limit of slowly varying refractive index and over short periods of time. Consider the following coordinate transformations:

$$\bar{x} = x \tag{42}$$

$$\bar{t} = \frac{t}{n(x)} \tag{43}$$

under which the sine-Gordon equation becomes

$$\frac{\partial^2\phi}{\partial\bar{x}^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial\bar{t}^2} + \bar{t}\,\frac{n'}{n}\left[-2\,\frac{\partial^2}{\partial\bar{t}\partial\bar{x}} + \frac{n'}{n}\frac{\partial}{\partial\bar{t}} + \bar{t}\,\frac{\partial^2}{\partial\bar{t}^2}\right]\phi = a \sin b\phi \tag{44}$$

in which $n' = dn/dx$. For the kink solution, $\partial^2\phi/\partial x^2 = a.b.O(1)$ and $(1/c^2)$ $\partial^2\phi/\partial t^2 = a.b.O(v_k^2/c^2)$. In the limit $[\bar{t}n'/n] << v_k/c^2$, we can ignore the bracket terms in (44) and obtain sine-Gordon equation in the $(\bar{x}, \bar{t})$ coordinates. The kink solution in the new coordinates reads

$$\phi(\bar{x}, \bar{t}) = \phi(x, t) = 4\tan^{-1}\exp[(ab)^{1/2}\gamma(v_0)(\bar{x} - v_0\bar{t})] \tag{45}$$

in which v_0 is a constant. Note that in the (x, t) coordinates $\bar{x} - v_0\bar{t} = x - [v_0/n(x)]t$ which corresponds to a varying kink velocity. The instantaneous position of the kink is obtained by solving the equation

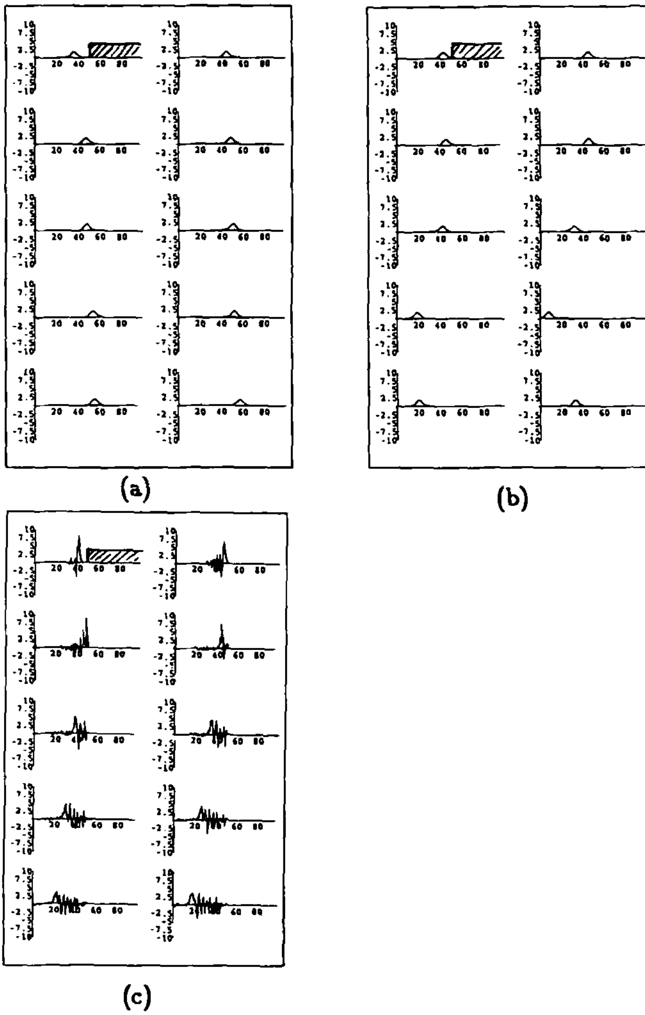

Fig. 2. Kink colliding with a potential step. v_k = 0.01: kink crosses over; v_k = 0.1: kink is reflected; v_k = 0.97: $k\bar{k}$ production together with low-amplitude waves.

$$n(x_k)x_k = v_0 t \tag{46}$$

for $x_k(t)$. The instantaneous acceleration of the kink is

$$a_k(x_k, v_k) = -\frac{2n'(x_k) + n''(x_k)x_k}{[n(x_k) + n'(x_k)x_k]^2} v_0 v_k \tag{47}$$

The potential acting on the kink as a result of varying refractive index of

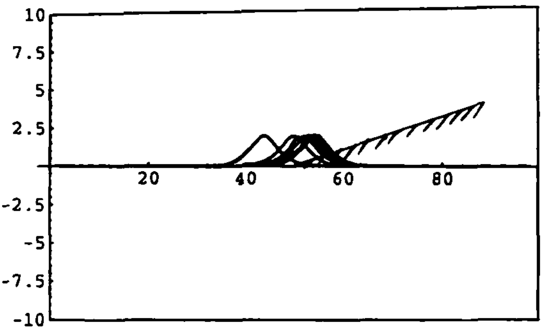

Fig. 3. Kink moving across a medium with linearly increasing refractive index.

the medium is clearly velocity dependent. The velocity dependence is linear in this approximation. This is in agreement with the numerical results.

3.2. Inhomogeneity of the Second Kind

This kind of inhomogeneity seems to be more interesting and viable to analytical description.

Equation (39) with a and b constants possesses kink solutions having rest energy

$$E_k(0) = 8\,\frac{a^{1/2}}{b^{3/2}} \tag{48}$$

and total energy

$$E_k(v_k) = \gamma(v_k)E_k(0) \tag{49}$$

Let us define a reference rest energy E_0 according to

$$E_0 = 8\,\frac{a_0^{1/2}}{b_0^{3/2}} \tag{50}$$

We can now write (49) in the form

$$E_k(v_k) = E_0 + T + U \tag{51}$$

where

$$U = E_k(0) - E_0 \tag{52}$$

That is, we have naturally decomposed the total kink energy in a homogeneous medium into three parts: rest energy (E_0), kinetic energy (T), and potential energy (U). From (48), (50), and (52), we have

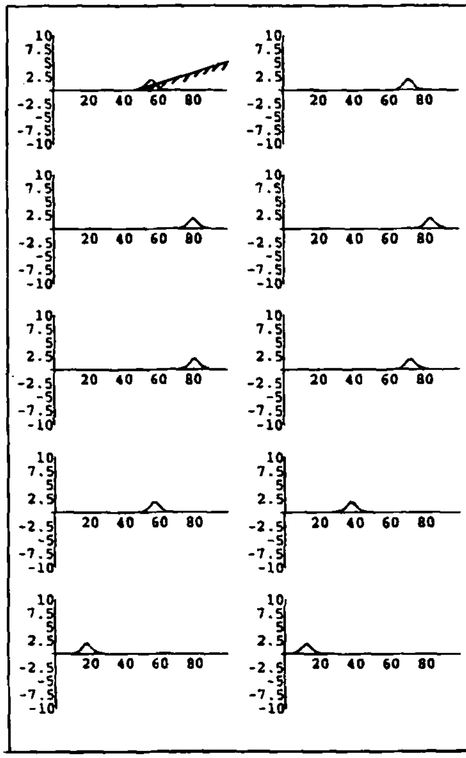

Fig. 4. Kink moving across a potential slope (second kind).

$$U = 8\,\frac{a^{1/2}}{b^{3/2}} - 8\,\frac{a_0^{1/2}}{b_0^{3/2}} = E_0\left(\left(\frac{a}{a_0}\right)^{1/2}\left(\frac{b_0}{b}\right)^{3/2} - 1\right) \tag{53}$$

We can now tentatively generalize this definition to the case $a = a(x)$ and $b = b(x)$. The kink dynamics can then be well described through the conventional prescription of classical relativistic dynamics, as long as the scale over which the potential varies appreciably is large compared with the size of the kink. The force field is conservative, and common energy conservation arguments for a massive particle hold.

Figure 4 shows the example of a kink moving against a potential slope. The kink returns to its initial position and velocity after a finite time. As a classical particle, the kink bypasses a potential barrier, as long as its kinetic energy is greater than the potential height (see Fig. 5). However, it is interesting to note that it does penetrate a potential barrier with $U > T$ if the barrier is thin enough. Figure 6 shows this classical 'tunneling effect.'

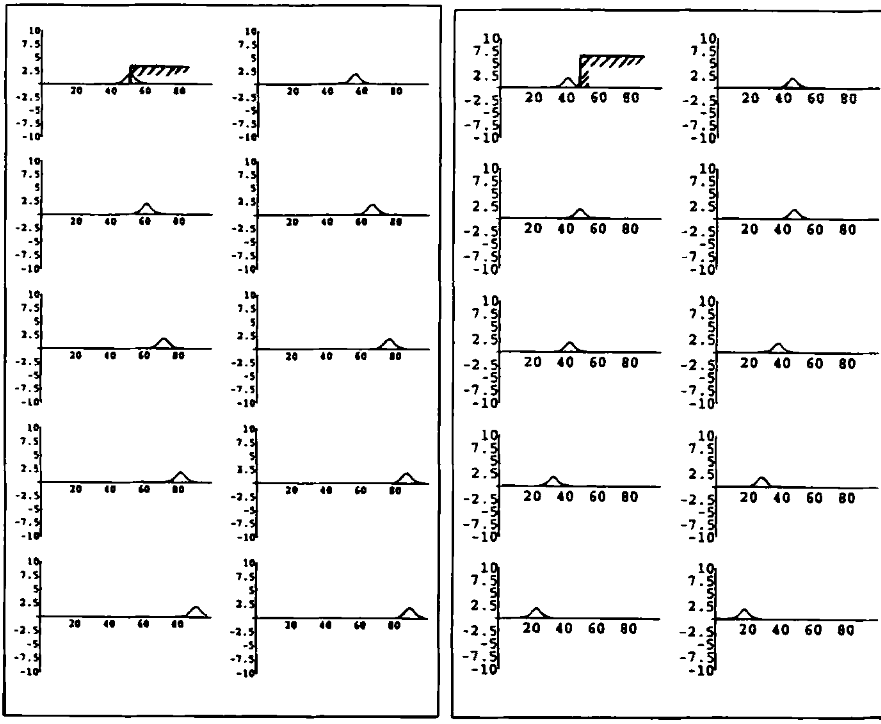

Fig. 5. (Left) kink passing over a potential step for $T > U$ and (right) being reflected for $T < U$.

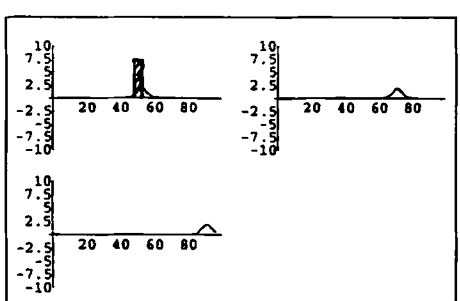

Fig. 6. Classical tunneling of a kink through a thin barrier with $T < U$.

This effect is a result of the wave aspect of the kink and does not contradict the particle aspect just described, because in this case the slowly varying assumption for the potential breaks down.

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